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Advanced Linear Algebra (MA 409)
Problem Sheet - 23

Gram-Schmidt Orthogonalization Process and Orthogonal Complements

1. Label the following statements as true or false.
   (a) The Gram Schmidt orthogonalization process allows us to construct an orthonormal set from an arbitrary set of vectors.
   (b) Every nonzero finite-dimensional inner product space has an orthonormal basis.
   (c) The orthogonal complement of any set is a subspace.
   (d) If $\{v_1, v_2, \ldots, v_n\}$ is a basis for an inner product space V, then for any $x \in V$ the scalars (x, v_i) are the Fourier coefficients of x.
   (e) An orthonormal basis must be an ordered basis.
   (f) Every orthogonal set is linearly independent.
   (g) Every orthonormal set is linearly independent.

2. In each part, apply the Gram Schmidt process to the given subset S of the inner product space V to obtain an orthogonal basis for $span(S)$.Then normalize the vectors in this basis to obtain an orthonormal basis β for $span(S)$, and compute the Fourier coefficients of the given vector relative to β.
   (a) $V = \mathbb{R}^3$, $S = \{(1,1,1), (0,1,1), (0,0,1)\}$, and $x = (1,0,1)$
   (b) $V = P_2(R)$ with the inner product $\langle f(x), g(x)\rangle = \int_0^1 f(t)g(t)dt$, $S = \{1, x, x^2\}$, and $h(x) = 1 + x$.
   (c) $V = span(S)$, where $S = \{(1, i, 0), (1-i, 2, 4i)\}$, and $x = (3+i, 4i, -4)$
   (d) $V = \mathbb{R}^4$, $S = \{(1,-2,-1,3), (3,6,3,-1), (1,4,2,8)\}$, and $x = (-1,2,1,1)$
   (e) $V = M_{2\times 2}(\mathbb{R})$, $S = \left\{ \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 11 & 4 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 4 & -12 \\ 3 & -16 \end{pmatrix} \right\}$, and $A = \begin{pmatrix} 8 & 6 \\ 25 & -13 \end{pmatrix}$
   (f) $V = span(S)$ with the inner product $\langle f, g\rangle = \int_0^\pi f(t)g(t)dt$, $S = \{\sin t, \cos t, 1, t\}$, and $h(t) = 2t + 1$
   (g) $V = \mathbb{C}^4$, $S = \{(-4, 3-2i, i, 1-4i), (-1-5i, 5-4i, -3+5i, 7-2i), (-27-i, -7-6i, -15+25i, -7-6i)\}$, and $x = (-13-7i, -12+3i, -39-11i, -26+5i)$
   (h) $V = M_{2\times 2}(\mathbb{C})$, $S = \left\{ \begin{pmatrix} -1+i & -i \\ 2-i & 1+3i \end{pmatrix}, \begin{pmatrix} -1-7i & -9-8i \\ 1+10i & -6-2i \end{pmatrix}, \begin{pmatrix} -11-132i & -34-31i \\ 7-126i & -71-5i \end{pmatrix} \right\}$, and $A = \begin{pmatrix} -7+5i & 3+18i \\ 9-6i & -3+7i \end{pmatrix}$

3. In $\mathbb{R}^2$, let

$$\beta = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right) \right\}.$$

   Find the Fourier coefficients of (3,4) relative to β.

4. Let $S = \{(1, 0, i), (1, 2, 1)\}$ in $\mathbb{C}^3$. Compute $S^\perp$.

5. Let $S_0 = \{x_0\}$, where x_0 is a nonzero vector in $\mathbb{R}^3$. Describe $S_0^\perp$ geometrically. Now suppose that $S = \{x_1, x_2\}$ is a linearly independent subset of R^3. Describe $S^\perp$ geometrically.

6. Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $x \notin W$, prove that there exists $y \in V$ such that $y \in W^\perp$, but $\langle x, y \rangle \neq 0$.

7. Let β be a basis for a subspace W of an inner product space V, and let $z \in V$. Prove that $z \in W^\perp$ if and only if $\langle z, v \rangle = 0$ for every $v \in \beta$.

8. Prove that if $\{w_1, w_2, \ldots w_n\}$ is an orthogonal set of nonzero vectors, then the vectors $v_1, v_2, \ldots, v_n$ derived from the Gram Schmidt process satisfy $v_i = w_i$ for $i = 1, 2, \ldots, n$.

   Hint : Use mathematical induction.

9. Let $W = span(\{(i, 0, 1)\})$ in $\mathbb{C}^3$. Find orthonormal bases for W and $W^\perp$.

10. Let W be a finite-dimensional subspace of an inner product space V. Prove that there exists a projection T on W along $W^\perp$ that satisfies $N(T) = W^\perp$. In addition, prove that $\|T(x)\| \leq \|x\|$ for all $x \in V$.

11. Let A be an $n \times n$ matrix with complex entries. Prove that $AA^* = I$ if and only if the rows of A form an orthonormal basis for C^n.

12. Prove that for any matrix $A \in M_{m\times n}(F)$, $(R(L_{A^*}))^\perp = N(L_A)$.

13. Let V be an inner product space, S and S_0 be subsets of V, and W be a finite-dimensional subspace of V. Prove the following results.

    (a) $S_0 \subseteq S$ implies that $S^\perp \subseteq S_0^\perp$.
    (b) $S \subseteq (S^\perp)^\perp$;so $span(S) \subseteq (S^\perp)^\perp$.
    (c) $W = (W^\perp)^\perp$.
    (d) $V = W \oplus W^\perp$.

14. Let W_1 and W_2 be subspaces of a finite-dimensional inner product space. Prove that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$ and $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$.

15. Let V be a finite-dimensional inner product space over F.

    (a) *Parseval's Identity*. Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V. For any $x, y \in V$ prove that

    $$\langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}.$$

    (b) Use (a) to prove that if β is an orthonormal basis for V with inner product $\langle \cdot, \cdot \rangle$, then for any $x, y \in V$

    $$\langle \phi_\beta(x), \phi_\beta(y) \rangle' = \langle |x|_\beta, |y|_\beta \rangle' = \langle x, y \rangle,$$

    where $\langle \cdot, \cdot \rangle'$ is the standard inner product on F^n.

16. (a) *Bessel's Inequality*. Let V be an inner product space, and let $S = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal subset of V. Prove that for any $x \in V$ we have

$$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i \rangle|^2,$$

(b) In the context of (a), prove that Bessel's inequality is an equality if and only if $x \in span(S)$.

17. Let T be a linear operator on an inner product space V. If $\langle T(x), y \rangle = 0$ for all $x, y \in V$, prove that $T = T_0$. In fact, prove this result if the equality holds for all x and y in some basis for V.

18. Let $V = C([-1,1])$. Suppose that W_e and W_o denote the subspaces of V consisting of the even and odd functions, respectively. Prove that $W_e^{\perp} = W_o$, where the inner product on V is defined by

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt.$$

19. In each of the following parts, find the orthogonal projection of the given vector on the given subspace W of the inner product space V.

(a) $V = \mathbb{R}^2$, $u = (2,6)$, and $W = \{(x,y) : y = 4x\}$.
(b) $V = \mathbb{R}^3$, $u = (2,1,3)$, and $W = \{(x,y,z) : x + 3y - 2z = 0\}$.
(c) $V = P(\mathbb{R})$ with the inner product $\langle f(x), g(x) \rangle = \int_0^1 f(t)g(t)dt$, $h(x) = 4 + 3x - 2x^2$, and $W = P_1(\mathbb{R})$.

20. In each part of the above Exercise, find the distance from the given vector to the subspace W.

21. Let $V = C([-1,1])$ with the inner product $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt$, and let W be the subspace $P_2(\mathbb{R})$, viewed as a space of functions. Use the orthonormal basis $\left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{8}}(3x^2 - 1)\right\}$ to compute the "best" (closest) second-degree polynomial approximation of the function $h(t) = e^t$ on the interval $[-1,1]$.

22. Let $V = C([0,1])$ with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$. Let W be the subspace spanned by the linearly independent set $\{t, \sqrt{t}\}$.

(a) Find an orthonormal basis for W.
(b) Let $h(t) = t^2$. Use the orthonormal basis obtained in (a) to obtain the "best" (closest) approximation of h in W.

23. Let V be the vector space of all sequences σ in F (where $F = \mathbb{R}$ or $F = \mathbb{C}$) such that $\sigma(n) \neq 0$ for only finitely many positive integers n. For $\sigma, \mu \in V$, we define $\langle \sigma, \mu \rangle = \sum_{n=1}^{\infty} \sigma(n)\overline{\mu(n)}$. Since all but a finite number of terms of the series are zero, the series converges.

(a) Prove that $\langle \cdot, \cdot \rangle$ is an inner product on V, and hence V is an inner product space.
(b) For each positive integer n, let e_n be the sequence defined by $e_n(k) = \delta_{n,k}$, where $\delta_{n,k}$ is the Kronecker delta. Prove that $\{e_1, e_2, \ldots\}$ is an orthonormal basis for V.
(c) Let $\sigma_n = e_1 + e_n$ and $W = span(\{\sigma_n : n \geq 2\}$.
  (i) Prove that $e_1 \notin W$, so $W \neq V$.
  (ii) Prove that $W^{\perp} = \{0\}$, and conclude that $W \neq (W^{\perp})^{\perp}$.

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